
Probability and Statistic (for CS Students)

Lecture 2: Counting and Combinatorics I

University of Mazandaran

Fall 1405

What is Counting?

An experiment in probability:



Counting: How many **outcomes** satisfy some **event**?

What is Counting?



Roll



What is Counting?



Roll



6

$\{1,2,3,4,5,6\}$

What is Counting?



Roll



6

$\{1,2,3,4,5,6\}$



Roll Even Only



What is Counting?



Roll



6

$\{1,2,3,4,5,6\}$



Roll Even Only



3

$\{2,4,6\}$

What is Counting?



Roll



6

$\{1,2,3,4,5,6\}$



Roll Even Only



3

$\{2,4,6\}$



Roll



What is Counting?



Roll



6

$\{1,2,3,4,5,6\}$



Roll Even Only



3

$\{2,4,6\}$



Roll



36

$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{1,5\}$	$\{1,6\}$
$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{2,6\}$
$\{3,1\}$	$\{3,2\}$	$\{3,3\}$	$\{3,4\}$	$\{3,5\}$	$\{3,6\}$
$\{4,1\}$	$\{4,2\}$	$\{4,3\}$	$\{4,4\}$	$\{4,5\}$	$\{4,6\}$
$\{5,1\}$	$\{5,2\}$	$\{5,3\}$	$\{5,4\}$	$\{5,5\}$	$\{5,6\}$
$\{6,1\}$	$\{6,2\}$	$\{6,3\}$	$\{6,4\}$	$\{6,5\}$	$\{6,6\}$

Step Rule (a.k.a. Product Rule) of Counting

In an experiment with two steps, where

the first step's outcomes are from set A with $|A| = m$, and

the second step's outcomes are from set B with $|B| = n$,

$|B|$ is unaffected by outcome of first step,

Then the number of total outcomes of the experiment is

$$|A||B| = mn$$

Two-step experiment

$\rightarrow A \rightarrow B$

Generalization of The Step Rule

In an experiment with R steps, where

the first step has n_1 possible outcomes, and

the second step has n_2 possible outcome, and

...

the R^{th} step has n_R possible outcome,

and the number of outcomes in the k^{th} step is unaffected by outcomes of the its previous steps,

Then the number of total outcomes of the experiment is

$$n_1 \times n_2 \times \cdots \times n_R.$$

Example: Choosing Password

You need to choose a password for a computer account. The rule is that the password must start with two lowercase letters (a to z) followed by one capital letter (A to Z) followed by three digits (0,1,...,9). Example of a valid password is

ejT301

What is the total number of possible passwords?

Example: Choosing Password

You need to choose a password for a computer account. The rule is that the password must start with two lowercase letters (a to z) followed by one capital letter (A to Z) followed by three digits (0,1,...,9). Example of a valid password is

ejT301

What is the total number of possible passwords?

Steps in choosing the password:

- ▶ choose a lowercase letter (26 possible outcomes)
- ▶ choose a lowercase letter (26 possible outcomes)
- ▶ choose a capital letter (26 possible outcomes)
- ▶ choose a digit (10 possible outcomes)
- ▶ choose a digit (10 possible outcomes)
- ▶ choose a digit (10 possible outcomes)

$$N = 26 \times 26 \times 26 \times 10 \times 10 \times 10$$

How Many Unique Images?

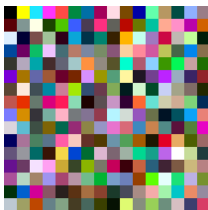
Hint 1: images are combination of pixels with different shades of RGB

Hint 2: each pixel is a combination of three colors R, G, and B

Hint 3: each color R, G, or B is coded using 8 bits



(a) 12 pixels



(b) 300 pixels



(c) 12 million pixels

How Many Unique Images?



Step 1: count # of possible values for each color R, G, and B

Step 2: count # of possible values for the first pixel

Step 3: count # of possible values for the second pixel

...

Step 12: count # of possible values for the 12th (last) pixel

$$N = \underbrace{256^3 \times 256^3 \times \cdots \times 256^3}_{12 \text{ times}}$$

How Many Unique Images?

Hint 1: images are combination of pixels with different shades of RGB

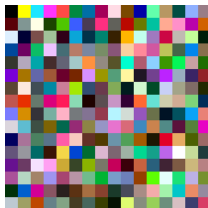
Hint 2: each pixel is a combination of three colors R, G, and B

Hint 3: each color R, G, or B is coded using 8 bits



(a) 12 pixels

$$\sim 10^{86}$$



(b) 300 pixels

$$\sim 10^{2167}$$



(c) 12 million pixels

$$\sim 10^{86696638}$$

How Many Unique Images?



Sum Rule of Counting

If the outcome of an experiment can be either from

set A with $|A| = m$, OR

set B with $|B| = n$,

iff there is no element that's in both sets, i.e., $A \text{ AND } B = \emptyset$,

Then the number of total outcomes of the experiment is

$$|A| + |B| = m + n$$

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A and B are called mutually exclusive

How Many Toys My Daughter Has?



(a) cats



(b) rabbits

How Many Toys My Daughter Has?



A More Serious Problem

Question: A 6-bit string is sent over a network. The valid set of strings recognized by the receiver must either start with “01” or end with “10”. How many such strings are there?

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Answer:

Set A: strings start with 01: $01_ _ _ _ \rightarrow 2^4 = 16$

Set B: strings end with 10: $_ _ _ _ 10 \rightarrow 2^4 = 16$

Set $A \cap B$: strings start with 01 and end with 10:
 $01_ _ 10 \rightarrow 2^2 = 4$

Total number: $16 + 16 - 4 = 28$

Or Rule of Counting (aka Inclusion/Exclusion)

If the outcome of an experiment can be either from

set A with $|A| = m$, OR

set B with $|B| = n$,

where A AND B *may not be empty*,

Then the number of total outcomes of the experiment is

$$|A| + |B| - |A \cap B|$$

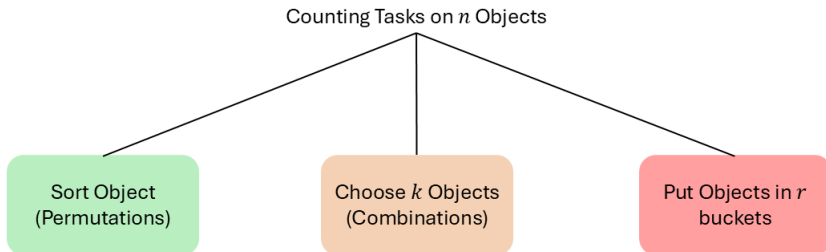
▶ Step rule of counting

- ▶ the experiment runs in r steps
- ▶ the first step has n_1 possible outcomes, and
- ▶ the second step has n_2 possible outcome, and
- ▶ ...
- ▶ the r^{th} step has n_r possible outcome,
- ▶ number of outcomes in the k^{th} step is unaffected by outcomes of the its previous steps,
- ▶ the number of outcomes of the experiment $n_1 \times n_2 \times \cdots \times n_r$

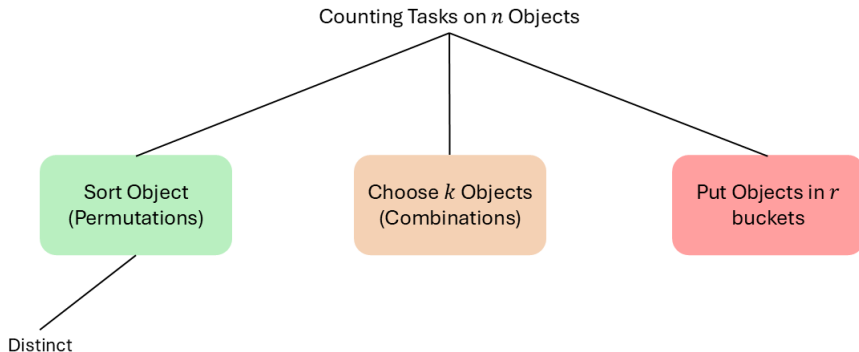
▶ Or rule of counting (for two outcomes)

- ▶ the outcome of an experiment can either be drawn from set A or set B
- ▶ the sets may potentially overlap
- ▶ the number of outcomes of the experiment $|A| + |B| - |A \cap B|$

Summary of Combinatorics



Summary of Combinatorics



How many different orderings of letters are possible for the following strings?

1. CHRIS

Orderings of Letters

- ▶ Step 1: choose 1st letter
- ▶ Step 2: choose 2nd letter
- ▶ Step 3: choose 3rd letter
- ▶ Step 4: choose 4th letter
- ▶ Step 5: choose 5th letter

Orderings of Letters

- ▶ Step 1: choose 1st letter → 5 options
- ▶ Step 2: choose 2nd letter → 4 options
- ▶ Step 3: choose 3rd letter → 3 options
- ▶ Step 4: choose 4th letter → 2 options
- ▶ Step 5: choose 5th letter → 1 option

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$$\implies N = 5 \times 4 \times 3 \times 2 \times 1$$

Orderings of Letters

chirs	crish	hicrs	hsirc	irchs	rcish	rschi	shirc
chisr	crshi	hicsr	hsrci	ircsh	rcshi	rscih	shrci
chris	crsih	hircs	hsric	irhcs	rcsih	rshci	shric
chrsi	cshir	hirsc	ichrs	irhsc	rhcis	rshic	sichr
chsir	csfri	hisrc	ichsr	irsch	rhcsi	rsich	sicrh
chsri	csihr	hisrc	icrhs	irshc	rhics	rsihc	sihcr
cihrs	csirh	hrcis	lcrsh	ischr	rhisc	rchir	sihrc
cihsr	csrhi	hrcsi	icshr	ischr	rhsci	rchri	sirch
cirsh	csrih	hrics	icsrh	ishcr	rhsic	rcihr	sirhc
cirsh	hcirs	hrisc	ihcrs	ishrc	richs	rcirh	srchi
cishr	hcisr	hrsci	ihcsr	isrch	ricsh	scrhi	srcih
cisrh	hcris	hrsic	ihrcs	isrhc	rihcs	scrih	srhci
crhis	hcrsi	hscir	ihrsc	lchis	rihsc	shcir	srhic
crhsi	hcsir	hscri	ihscr	rchsi	risch	shcri	srich
crihs	hcsir	hsicr	ihsrc	rcihs	rishc	shicr	srihc

Python code

```
import itertools

def main():
    letters=['c', 'h', 'i', 'r', 's']
    perms=set(itertools.permutations(letters))

    for p in perms:
        print(p)
```

Ranking of Students

Assume a probability class with 6 girls and 4 boys. After the final exam, students are ranked according to their performance. Assuming no two students obtain same score, how many different rankings are possible?

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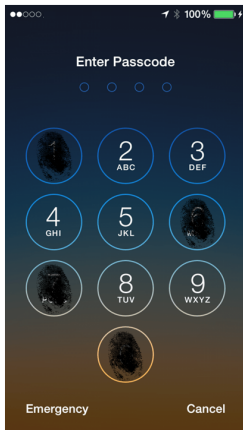
$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

A **permutation** is an **ordered** arrangement of distinct objects

The number of unique orderings (**permutations**) of n distinct objects is

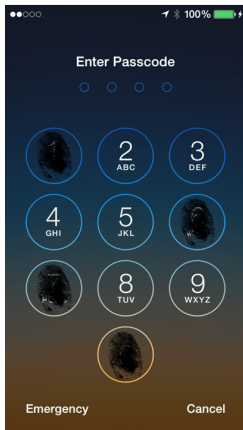
$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

Passcodes



A smartphone has 4-digit passcode. Suppose there are 4 smudges over 4 digits on the screen. How many distinct passcodes are possible?

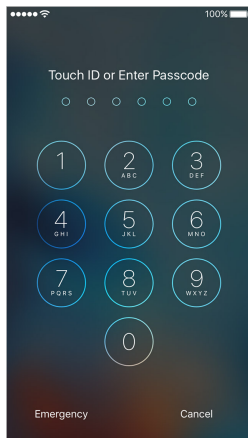
Passcodes



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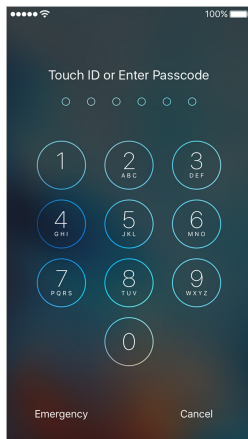
$$\text{Total} = 4 \times 3 \times 2 \times 1 = 24$$

Passcodes



How many unique 6-digit passcodes are possible if a phone password is some ordered subset of any distinct 6 digits?

Passcodes



How many unique 6-digit passcodes are possible if a phone password is some ordered subset of any distinct 6 digits?

$$\text{Total} = 10 \times 9 \times 8 \times 7 \times 6 \times 5$$

$$\frac{10!}{4!} = 151200$$

Summary of Combinatorics

Counting Tasks on n Objects

Sort Object
(Permutations)

Choose k Objects
(Combinations)

Put Objects in r
buckets

Distinct

$n!$