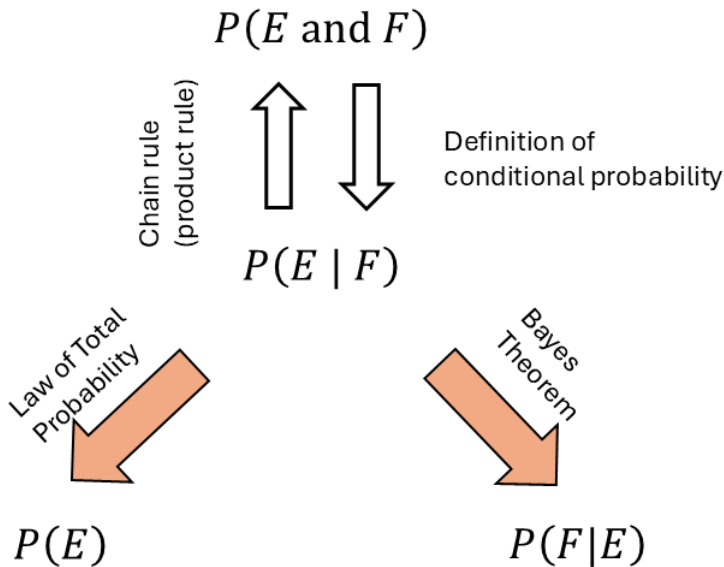

Probability and Statistic (for CS Students)

Lecture 7: Independence

University of Mazandaran

Fall 1405

Recap: Conditional Probability, Bayes, and Chain Rule



Recap: Bayes' Theorem

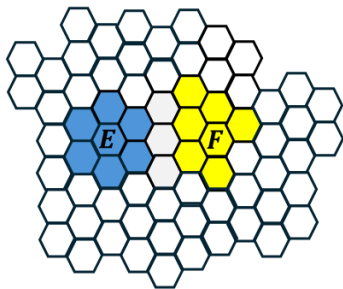
Theorem: For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}$$

Expanded form:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)}$$

Today's Learning Goal



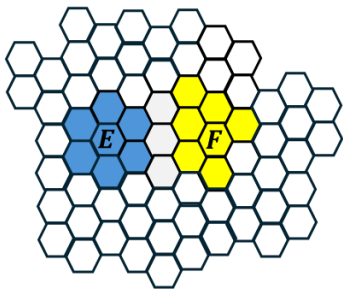
Mutually Exclusive

$$P(A \text{ and } B) = 0$$

Makes **OR** easy:

$$P(A \text{ or } B) = P(A) + P(B)$$

Today's Learning Goal



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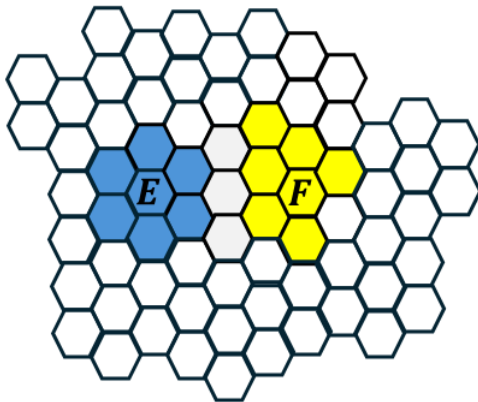
Independent

$$P(A) = P(A|B)$$

Makes **AND** easy:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

OR With Mutually Exclusive Events Recap

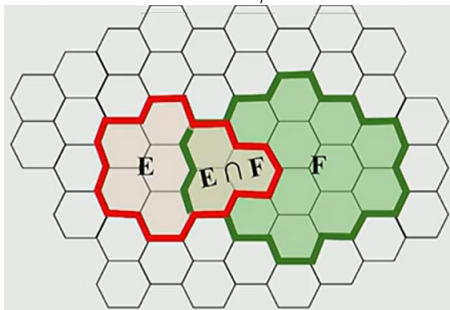


If events are mutually exclusive, probability of OR is simple:

$$P(E \cup F) = P(E) + P(F)$$

OR Without Mutually Exclusive Events

a.k.a. inclusion/exclusion

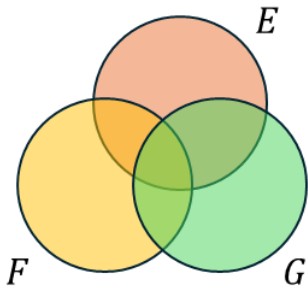


$$P(E \cup F) = P(E) + P(F) - P(E, F)$$

OR Without Mutually Exclusive Events

More than two sets?

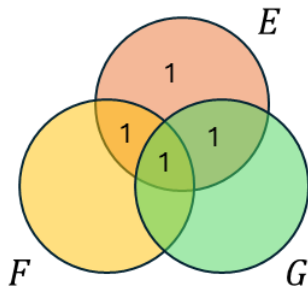
$$P(E \cup F \cup G) =$$



OR Without Mutually Exclusive Events

More than two sets?

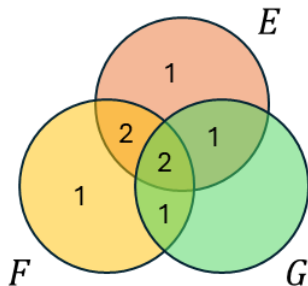
$$P(E \cup F \cup G) = P(E)$$



OR Without Mutually Exclusive Events

More than two sets?

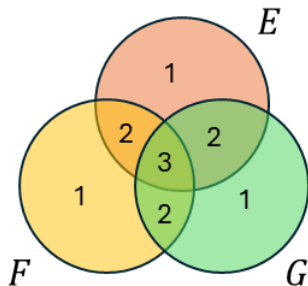
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OR Without Mutually Exclusive Events

More than two sets?

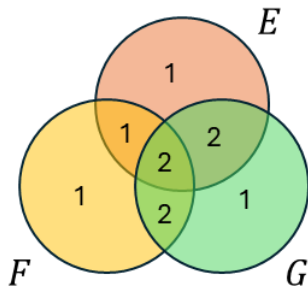
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OR Without Mutually Exclusive Events

More than two sets?

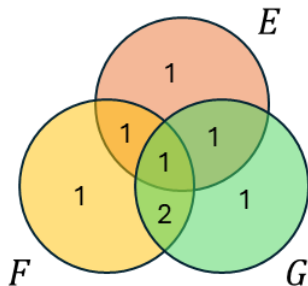
$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E, F)$$



OR Without Mutually Exclusive Events

More than two sets?

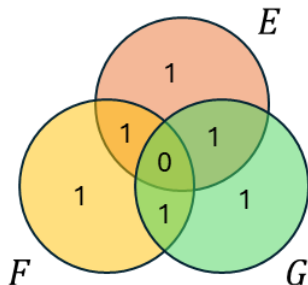
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OR Without Mutually Exclusive Events

More than two sets?

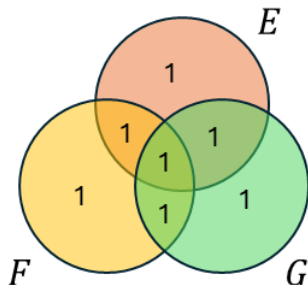
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OR Without Mutually Exclusive Events

More than two sets?

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E, F) - P(E, G) - P(F, G) + P(E, F, G)$$



General Inclusion/Exclusion

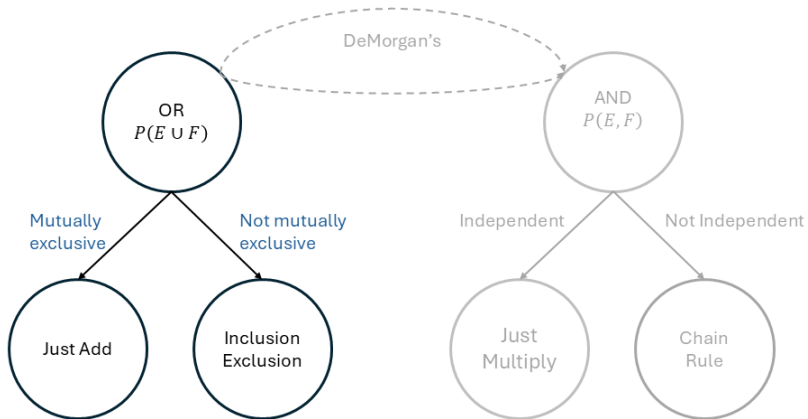
$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n (-1)^{i+1} Y_i$$

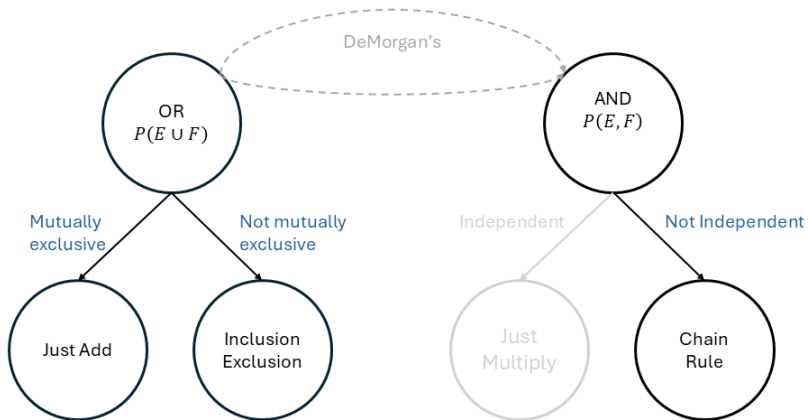
$$Y_1 = \text{Sum of all events on their own} \quad \sum_i P(E_i)$$

$$Y_2 = \text{Sum of all pairs of events} \quad \sum_{i,j, i \neq j} P(E_i \cap E_j)$$

$$Y_3 = \text{Sum of all triples of events} \quad \sum_i P(E_i \cap E_j \cap E_k)$$

and so on





Two events A and B are called independent if:

$$P(A) = P(A|B)$$

knowing that event B happened, doesn't change our belief that A will happen.

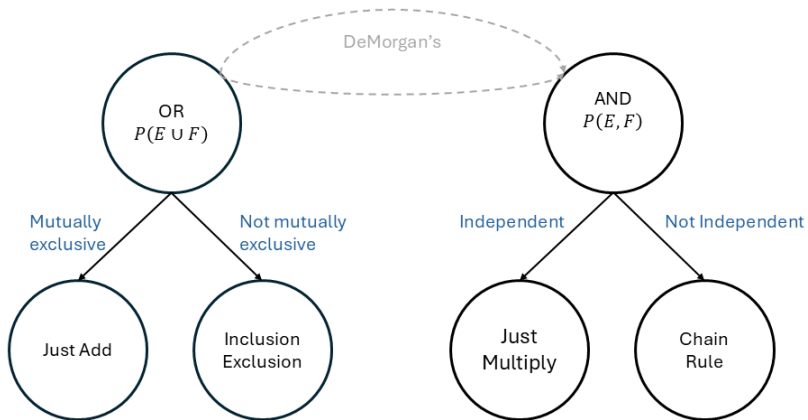
Otherwise, they are called dependent events.

Alternative Definition of Independence

$$\begin{aligned} P(A, B) &= P(A) \cdot P(B|A) && \text{Chain rule} \\ &= P(A) \cdot P(B) && \text{Since } B \text{ is independent of } A \end{aligned}$$

If you show this is true, you have proved the two events are independent!

If events are *independent*,
probability of *AND* is easy.



Roll two 6-sided dice, yielding values D_1 and D_2

- ▶ Let E be event: $D_1 = 1$
- ▶ Let F be event: $D_2 = 1$

Are E and F (in)dependent?

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Are E and F (in)dependent?

- ▶ $P(E) = 1/6$, $P(F) = 1/6$, $P(E, F) = 1/36$
- ▶ $P(E, F) = P(E)P(F) \rightarrow E$ and F independent

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Let G be event: $D_1 + D_2 = 5 \quad \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

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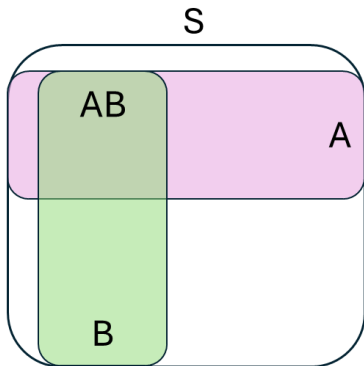
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Are E and G (in)dependent?

- ▶ $P(E) = 1/6$, $P(G) = 4/36$, $P(E, G) = 1/36$
- ▶ $P(E, G) \neq P(E)P(G) \rightarrow E$ and G dependent

What does independence look like!?

Independence



Independence Definition 1:

$$P(A, B) = P(A)P(B)$$

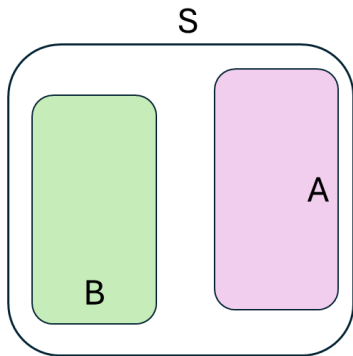
$$\frac{|A, B|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

Independence Definition 2:

$$P(A|B) = P(A)$$

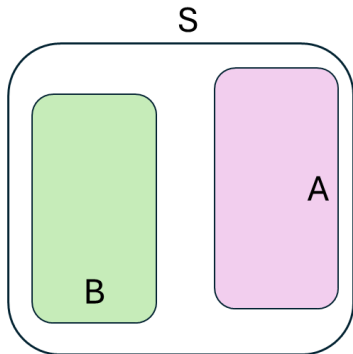
$$\frac{|A, B|}{|B|} = \frac{|A|}{|S|}$$

How about this one?



Independence

How about this one?



Independence Definition 1:

$$P(A, B) = P(A)P(B)$$

$$\frac{|A, B|}{|S|} = \frac{|A|}{|S|} \times \frac{|B|}{|S|}$$

A red arrow points from the numerator $|A, B|$ to the superscript 0 in the fraction $\frac{|A|}{|S|}$, indicating that the intersection of A and B is empty.

Not independent!

Given independent events A and B , prove that A and B^c are independent.

Given independent events A and B , prove that A and B^c are independent.

Want to show that $P(A, B^c) = P(A)P(B^c)$

$$\begin{aligned}P(A, B^c) &= P(A) - P(A, B) \quad \text{By Law of Total Prob.} \\&= P(A) - P(A)P(B) \quad \text{By independence} \\&= P(A)[1 - P(B)] \quad \text{Factoring} \\&= P(A)P(B^c) \quad \text{Since } P(B) + P(B^c) = 1\end{aligned}$$

So, if A and B are independent, A and B^c are also independent!

General definition of Independence:

Events $E_1, E_2, \dots, E_n$ are independent if for every subset with r elements (where $r \leq n$) it holds that:

$$P(E_1, E_2, \dots, E_r) = P(E_1) P(E_2) \cdots P(E_r)$$

Example: outcomes of n separate flips of a coin are all independent of one another

Two Dice

Roll two 6-sided dice, yielding values D_1 and D_2

- ▶ Let E be event $D_1 = 1$
- ▶ Let F be event $D_2 = 6$
- ▶ Let G be event $D_1 + D_2 = 7$

Are E and F independent?

Are E and G independent?

Are F and G independent?

Are E , F and G independent?

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Are F and G independent? Yes

- ▶ $P(F) = 1/6$, $P(G) = 1/6$, $P(F, G) = 1/36$

Are E , F and G independent? No

- ▶ $P(E, F, G) = 1/36 \neq 1/216 = (1/6)(1/6)(1/6)$

Example: Children as Independent Trials

Children's hair type is determined by their parents' genes. Gene a defines curly hair and children will only have curly hair if they get gene a from both parents. Two parents both have an (A, a) gene pair.

- ▶ each parent will pass on one of their genes (each gene equally likely) to a child
- ▶ the probability of any single child having curly hair is 0.25, independent of other siblings.
- ▶ there are three children.

What is the probability that all three children have curly hair?

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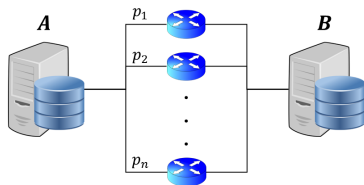
$$\begin{aligned} P(E_1, E_2, E_3) &= \\ P(E_1) P(E_2|E_1) P(E_3|E_1, E_2) &= \\ P(E_1) P(E_2) P(E_3) \end{aligned}$$

Network Reliability

Consider the following parallel network:

- ▶ n independent routers, each with probability p_i of functioning (where $1 \leq i \leq n$)
- ▶ E = functional path from A to B exists.

What is $P(E)$?



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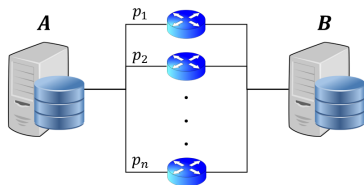
What is $P(E)$?

$$P(E) = P(\geq 1 \text{ one router works})$$

$$1 - P(\text{all routers fail})$$

$$1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n)$$

$$1 - \prod_{i=1}^n (1 - p_i)$$



Conditional Independence

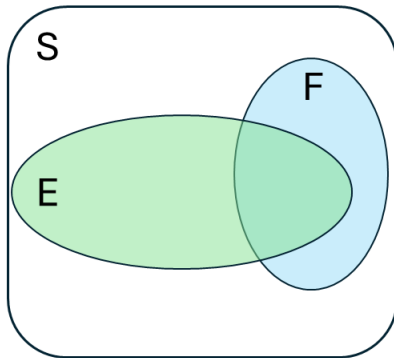
- ▶ Two events E and F are independent if

$$P(E, F) = P(E)P(F) \quad \text{or} \quad P(E|F) = P(E)$$

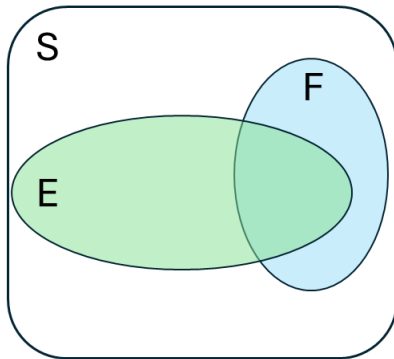
- ▶ Two events E and F are conditionally independent given an event C with $P(C) > 0$ if:

$$P(E, F|C) = P(E|C)P(F|C) \quad \text{or} \quad P(E|F, C) = P(E|C)$$

Conditional Independence

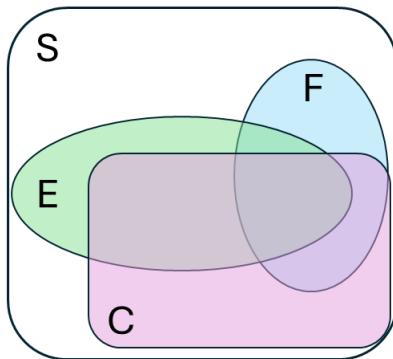


Conditional Independence



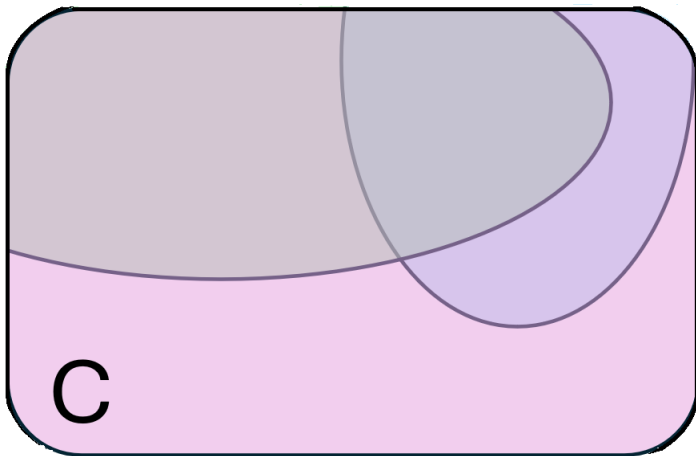
$$P(E, F) = P(E)P(F)$$

Conditional Independence



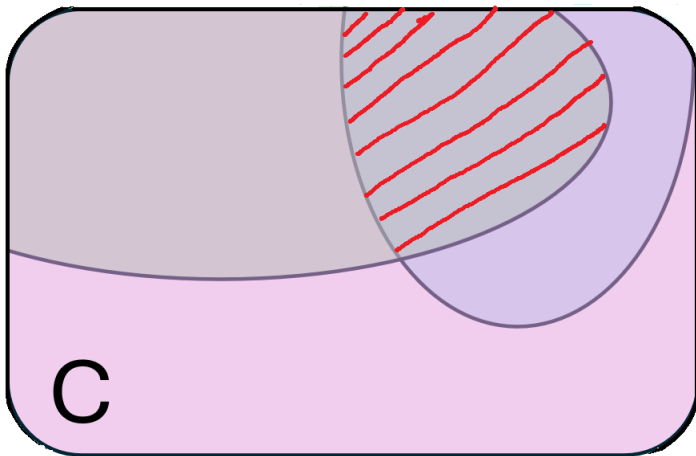
$$P(E, F | C) = P(E | C) P(F | C)$$

Conditional Independence



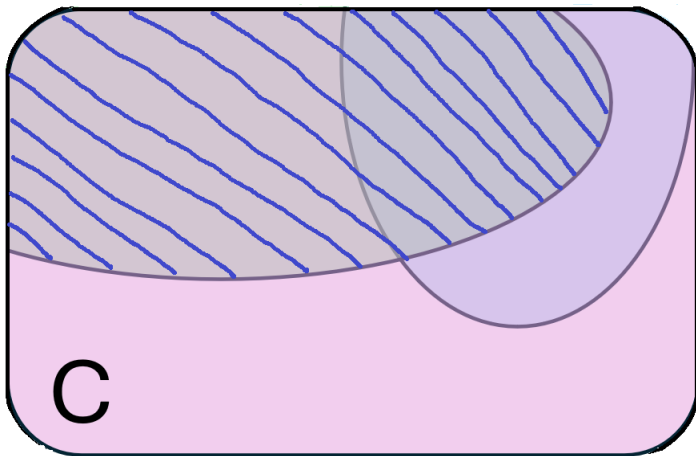
$$P(A \mid C) = P(A \mid B \mid C) P(B \mid C)$$

Conditional Independence



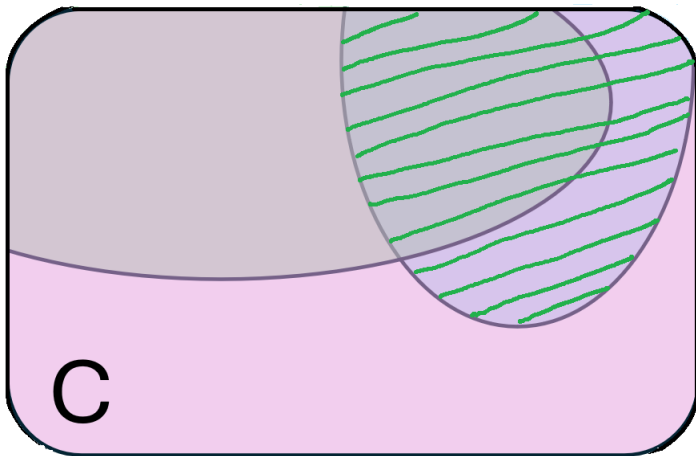
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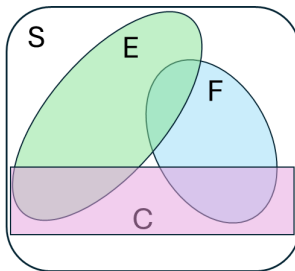


$$P(E, F|C) = P(E|C) P(F|C)$$

Example

Given the following Venn diagram, which statements is correct?
 E and F are

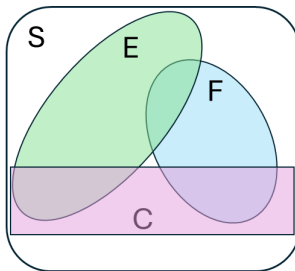
- ▶ independent given C
- ▶ independent
- ▶ dependent given C



Example

Given the following Venn diagram, which statements is correct?
 E and F are

- ▶ independent given C
- ▶ independent
- ▶ **dependent given C**



Example

In previous example, assume E and F be events that two people (living in the same apartment) arrive home on time.

- ▶ Probabilities E and F are independent
- ▶ Y is the event that a snow storm occurs
- ▶ Probabilities $E|Y$ and $F|Y$ are dependent (why?)

Example

In previous example, assume E and F be events that two people (living in the same apartment) arrive home on time.

- ▶ Probabilities E and F are independent
- ▶ Y is the event that a snow storm occurs
- ▶ Probabilities $E|Y$ and $F|Y$ are dependent (why?)

Knowing delay in E tells you that F will be delayed!

In the conditional paradigm, the **Bayes' formula** is preserved.

$$P(A|B, E) = \frac{P(B|A, E) P(A|E)}{P(B|E)}$$

E and F are conditionally independent given C . What can we say about their independence in general (without conditioning on C)?

E and F are conditionally independent given C . What can we say about their independence in general (without conditioning on C)?

Nothing!

Most concepts defined for probability can also be extended to conditional probability



Independence relationships can change with conditioning.

If E and F are independent, that does not mean they will still be independent given another event G .

What is the probability that a user will watch "Age of Samurai"?

$$P(E_1)$$



$$P(E_1) = \lim_{n \rightarrow \infty} \frac{\text{count}(E_1)}{n} \approx \frac{\# \text{ people who watched the series}}{\# \text{ people on Filimo}}$$

What is the probability that a user will watch "Age of Samurai" (E_2), given they watched "Shogun" (E_1)?

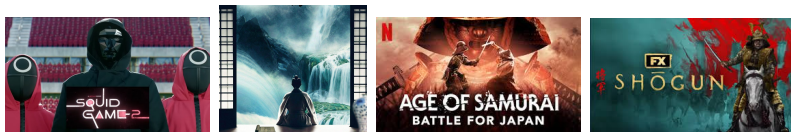
$$P(E_2|E_1)$$



$$P(E_2|E_1) = \frac{P(E_2, E_1)}{P(E_1)} = \frac{\# \text{ people who watched both}}{\# \text{ people who watched } E_1}$$

Conditioning on a set of movies!

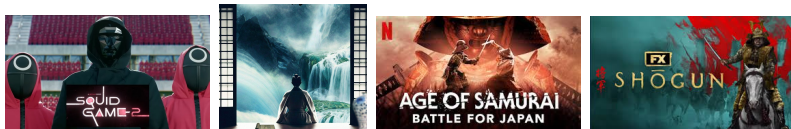
Each event corresponds to liking a particular movie



$$P(E_4 | E_1, E_2, E_3)$$

Conditioning on a set of movies!

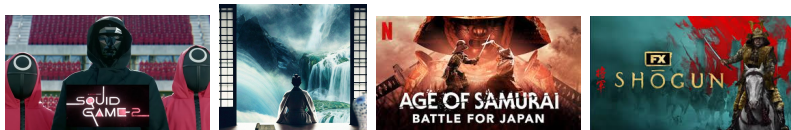
Each event corresponds to liking a particular movie



$$P(E_4|E_1, E_2, E_3) = \frac{P(E_1, E_2, E_3, E_4)}{P(E_1, E_2, E_3)}$$

Conditioning on a set of movies!

Each event corresponds to liking a particular movie



$$P(E_4|E_1, E_2, E_3) = \frac{P(E_1, E_2, E_3, E_4)}{P(E_1, E_2, E_3)}$$

PROBLEM!

What is the probability that a user watches four particular movies?

- ▶ There are 100,000 movies on Filimo
- ▶ The user watches 30 random titles
- ▶ E = movies watched include the given four.

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- ▶ There are 100,000 movies on Filimo
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- ▶ E = movies watched include the given four.

Solution:

$$P(E) = \frac{\binom{4}{4} \binom{99996}{26}}{\binom{100000}{30}} \approx 6.57 \times 10^{-15}$$

K : Event that a user likes east-Asian movies



It could be that E_1, E_2, E_3 , and E_4 are conditionally independent given K

K : Event that a user Likes east-Asian movies



It could be that E_1, E_2, E_3 , and E_4 are conditionally independent given K

$$P(E_4|E_1, E_2, E_3, K) = P(E_4|K)$$

E and F are dependent,
does not mean they will be dependent when another event
is observed

E and F are independent,
does not mean they will be independent when another
event is observed