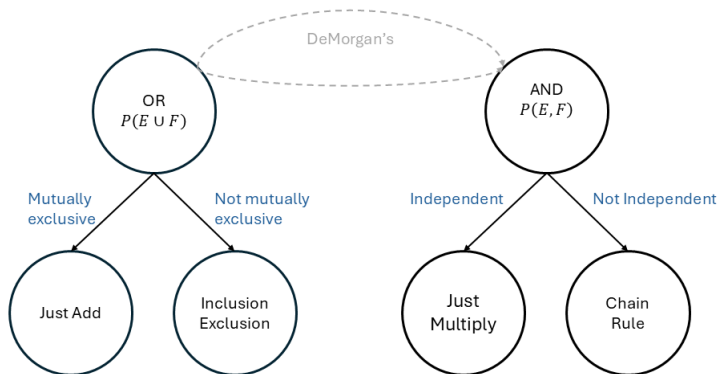

Probability and Statistic (for CS Students)

Lecture 8: Random Variables

University of Mazandaran

Fall 1405

Recap: Probability of Or and And



Recap: Conditional Probability

Two events A and B are **conditionally independent** given an event C with $P(C) > 0$ if:

$$P(A, B|C) = P(A|C) P(B|C)$$

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Equivalently:

$$P(A|B, C) = P(A|C)$$

The Most Important Probability Question

Say, a coin comes up heads with probability p

- ▶ Flip the coin n times
- ▶ Each coin flip is an **independent** trial
- ▶ What is the probability of exactly k heads?

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Finally: The probability of exactly $k = 4$ flips are heads

$$\binom{n}{k} p^k (1 - p)^{n-k}$$

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- ▶ $P(Y = 3) = 1/8$ $[H, H, H]$
- ▶ $P(Y \geq 3) = 0$

It is confusing that both R.V.s and events use the same notation

Consider 5 coin flips, each independently come up heads with probability p

► Recall:

$$P(2 \text{ heads}) = \binom{5}{2} p^2 (1-p)^3$$

$$P(3 \text{ heads}) = \binom{5}{3} p^3 (1-p)^2$$

► Y = number of "heads" on 5 flips

$$Y \in \{0, 1, 2, \dots, 5\}$$

$$P(Y = k) = \binom{5}{k} p^k (1-p)^{5-k}$$

- ▶ Probability Mass Function (PMF): $P(X = a)$
- ▶ Expectation: $E[X]$
- ▶ Variance: $\text{Var}(X)$

Probability Mass Function

The relationship between values a r.v. can take on, and the corresponding probability, is a *function*!

Probability Mass Function

Let Y be a random variable

Y

For example, Y is the number of heads in 5 coin flips

$$Y = 2$$

It is an event when it takes on a value

For example, Y is the number of heads in 5 coin flips

Any Boolean operation applies to a
R.V. can make it an event

Probability Mass Function

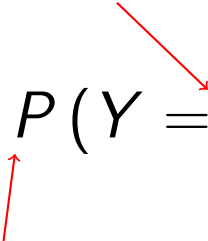
If this is a number


$$P(Y = 2)$$

For example, Y is the number of heads in 5 coin flips

Probability Mass Function

If this is a number


$$P(Y = 2)$$

Then, this is a number (between 0 and 1)

For example, Y is the number of heads in 5 coin flips

Probability Mass Function

If this is a variable


$$P(Y = k)$$

For example, Y is the number of heads in 5 coin flips

Probability Mass Function

If this is a variable


$$P(Y = k)$$



Then, this is a function

For example, Y is the number of heads in 5 coin flips

If a random variable is discrete, we call this function
Probability Mass Function

Let X be a random variable that represents the result of a **single die roll**. X can take on the values $\{1, 2, 3, 4, 5, 6\}$

$$P(X = x)$$

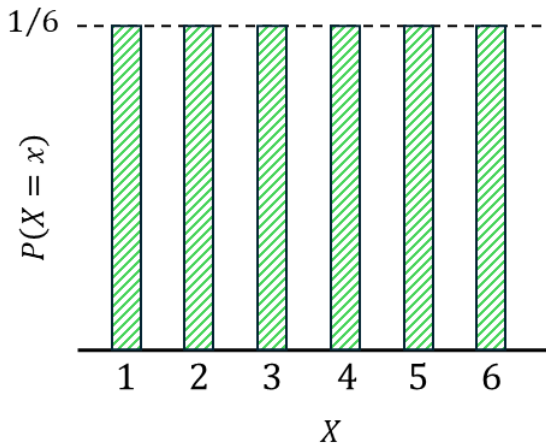
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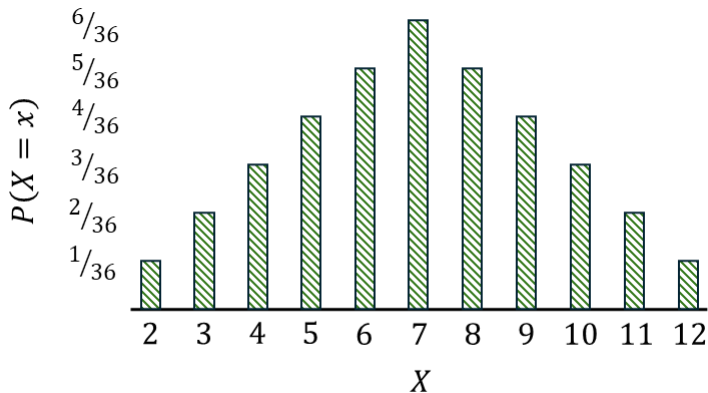
$p(x)$ a shorthand notation for the PMF

$p_x(x)$ another shorthand notation for the PMF

PMF For The Outcome of a Die Roll



PMF For The Sum of Two Dice Rolls



$$P(X = x) = \begin{cases} \frac{x-1}{36} & \text{if } x \in \mathbb{Z}, 1 \leq x \leq 6 \\ \frac{13-x}{36} & \text{if } x \in \mathbb{Z}, 7 \leq x \leq 12 \\ 0 & \text{else} \end{cases}$$

$$\sum_{\text{all } k} P(Y = k) = ?$$

$$\sum_{\text{all } k} P(Y = k) = 1$$

Expectation

The Expected Values for a discrete random variable X is defined as:

$$E[X] = \sum_{x:p(x)>0} x.P(X=x)$$

Note: sum over all values of x that have $p(x) > 0$

Expected value also called: **Mean**, Expectation, **Weighted Average**, **Center of Mass**, 1st Moment

Roll a 6-sided die. X is outcome of roll

$$\begin{aligned} \blacktriangleright P(X=1) &= P(X=2) = P(X=3) = P(X=4) = \\ &P(X=5) = P(X=6) = 1/6 \end{aligned}$$

$$E[X] = 1 \left(\frac{1}{6} \right) + 2 \left(\frac{1}{6} \right) + 3 \left(\frac{1}{6} \right) + 4 \left(\frac{1}{6} \right) + 5 \left(\frac{1}{6} \right) + 6 \left(\frac{1}{6} \right)$$

Mark Twain: "There are three kinds of lies: lies, damned lies, and statistics"

Schools has 3 classes with 5, 10, and 150 students

Randomly choose a class with equal probability

X = size of chosen class

What is $E[X]$?

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►
$$E[X] = 5 \left(\frac{1}{3}\right) + 10 \left(\frac{1}{3}\right) + 150 \left(\frac{1}{3}\right) = 55$$

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$$\blacktriangleright E[Y] = 5 \left(\frac{5}{165} \right) + 10 \left(\frac{10}{165} \right) + 150 \left(\frac{150}{165} \right) \approx 137$$

Linearity:

$$E[aX + b] = aE[X] + b$$

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$$\begin{aligned} E[aX + b] &= \sum_x (aX + b) p(x) \\ &= \sum_x (aXp(x) + bp(x)) \\ &= \sum_x aXp(x) + \sum_x bp(x) \\ &= a \sum_x Xp(x) + b \sum_x p(x) \\ &= aE[X] + b \end{aligned}$$

Properties of Expectation

Linearity:

$$E[aX + b] = aE[X] + b$$

Expectation of a sum is the sum of expectations:

$$E[X + Y] = E[X] + E[Y]$$

Properties of Expectation

$$E[X + Y] = E[X] + E[Y]$$

X	Y	$X + Y$
3	4	7
2	2	4
6	8	14
10	23	33
1	-3	-2
1	0	1
5	9	14
4	1	5

$$E[\ddot{X}] + E[\ddot{Y}] = E[\ddot{X + Y}]$$

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Unconscious Statistician:

$$E[g(X)] = \sum_x g(x) p(x)$$

Help us improve the course quality by giving us
feedback [here](#)



Probability for CS Students Feedback Form

B I U  

This form is intended to collect your feedback on how to improve the teaching, contents, students participation, etc. for the "Probability for CS Students" course.

What part of the course is your feedback about? (examples are teaching style, contents, teacher's behavior, etc.) *

Short-answer text

What is your feedback or suggestion to improve the course? *

Long-answer text

(optional) How can we reach you for follow-up question?

Short-answer text